THE ORIGIN OF HALF-INTEGRAL SPIN IN A DISCRETE PHYSICAL SPACE

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1. Introduction. The writers of several papers in this book—Atkin, Bohm, Penrose, von Weizsäcker—have made reference to a view of physical space in which space is defined in terms of a finite number of points and in which a rule is postulated for constructing new points. I shall call a physical space of this kind *constructive*, by analogy with the constructivist mathematics of Brouwer.[1]

I believe that a constructive theory of physical space is potentially capable of resolving the clash between the continuum aspect and the discrete aspect of quantum physics, because to introduce it is a sufficiently radical change to eliminate our reliance on classical concepts. These have to be separately defined in terms of the new approach.

The basis for the present paper is a constructive theory[2] in which a construction rule for new points generates a hierarchy. The points at a given level of this hierarchy replace the quantum-mechanical eigenvectors, and an order-preserving mapping between the levels corresponds to a physically significant event. In a short paper in Part VI of this book, I argue that in a constructive space a sufficient condition to be able to define an operationally realistic concept of particle trajectory is that the points in the space be ordered, and therefore for the purpose of the present paper I shall summarize only so much of the constructive theory as is necessary to define an ordered set of points.

I shall then deduce the existence of binary vectors having the formal structure of half-integral spin in relation to rotation. I shall also compare the conventional and the discrete–constructive theories in respect of the extent to which they constitute an explanation of the half-integral spin.

2. The construction algorithm. I consider a set S of abstract *events* upon which a basic operation *discrimination* is defined. When two entities in S, A and B, occur in an interactive relationship together, one of them, say A, can discriminate between the two cases:

(1) that B is identical with A;
(2) that B is not identical with A.

Further, the result of this discrimination operation is also an event, C, in the set, and $C \neq A$, $C \neq B$; the set has therefore a construction process built in.

It has been shown by Amson, Parker-Rhodes and the writer (2) that if we wish to represent the entities in this set by ordered sets of binary symbols, 0 and 1, then the discrimination operation has to be Boole's original operation of addition, sometimes called *symmetric difference*. In an arithmetic context (which at present we are not in) this is addition mod 2. This operation is represented by the table

	0	0
0	0	1
1	1	0

It was also shown, by introducing a concept of closure under the discrimination operation, that the discriminable entities in the set are the linearly independent sets of vectors, excluding the null vector, of which there are $2^n - 1$, where n is the cardinal of the set of entities generated. This set of sets of entities is made the elements of a new *level* in which discrimination can again take place. Proceeding as before we get $2^{(2n-1)} - 1$ of these discriminated entities at the level after, and so on.

The discrimination process requires that the vectors, or sets of binary units upon which the discrimination process acts, be ordered. This requirement is satisfied as an axiom in a hierarchy whose simplest level consists of the vectors 0, 1 themselves. Henceforward I shall only discuss this kind of hierarchy. Since discrimination operations can take place at different levels and since the existence of entities at the different levels depend upon those at previous levels, it follows that a *record* of what entities exist has to be kept, independently of the set of entities. When n entities exist, this record would require n binary units in the memory, and since they have to be ordered the number of units becomes n^2. For the case $n = 2$, which we are considering, we therefore have our independent record system increasing as the sequence

$$2, 4, 16, \ldots$$

as the higher levels are generated.

Obviously, a whole variety of ways exists in which the discrimination processes taking place in the hierarchy can be ordered or mapped onto the record or memory system. Physically significant applications of the hierarchy will be those that preserve some interesting structure at the different levels. The first properties of the hierarchy were discovered by Parker-Rhodes (see reference (2)) who related structures at different levels in the following way. He represented any given discrimination operation as a matrix transform, and then regarded the next level as constituted of vectors of order n^2 constructed from the $n \times n$ matrices used in the transform by any invariant rule (such as by putting all columns in the matrix end to end). If we write the discrimination process

$$AB \to C,$$

for A, B, C, column vectors of order n, then there will be a class of $n \times n$ matrices X, Y ... such that

$$A[X] \to C$$

etc. for a given B. This class can be used to define an element at the new level by writing each of the $n \times n$ matrices as n^2-vectors as I have described and by forming the union of the set under the discrimination operation. Such a method of construction of a new level preserves some type of order between the levels which is potentially capable of physical interpretation, and the matrix transform provides the record system which I have already argued to be necessary in the construction of a hierarchy and which increases in size by squaring of the existing record space at each change of level.

The method of level construction which I have just described contains several arbitrary features, and in fact the level to level relationship that it defines is not easily interpretable in detail physically. In a paper now in print (see reference (6)), R. H. Atkin and I define the set of n elements at any given level as the sum of the $n+1$, n, $n-1$, ..., simplicial complexes, and the physical interpretation of the individual complexes may then be considered, and the algebra of order-preserving mappings used upon the concept of levels in a hierarchy in a natural way. However, the only detailed physical problem to be discussed in this paper—that of the origin of spin—can be treated using the matrix method which I have outlined.

It is also possible to see, in spite of the arbitrary features of the matrix method of level construction, that an upper bound to the size of each level exists and that this bound is independent of the particular choice of relation between levels. The upper bound corresponds to the case when every possible discrimination has been made, and we immediately get the sequences

Level	0	1	2	3	4	5
Maximum number of discriminable quantities		3	7	127	$\sim 10^{38}$	STOP
Cumulative sums		3	10	137	$\sim 10^{38}$	STOP
Available space in record system	2	4	16	256	STOP	

The existence of the cumulative sums, with their correspondence with the reciprocals of the characteristic coupling constants of the physical fields, gave the first indication of possible physical significance for the hierarchy. It is recognized in current physics that the coupling constants—being pure numbers—have a unique status in that their values determine the relative scale of different types of physical phenomenon. There is a history—dating especially from the work of Eddington—of development of a point of view which attaches significance to these dimensionless constants, as constraints, first upon the possible values of the natural atomic and cosmological constants of which they are ratios, and hence on all measurements. This view would be able to explain the unique status universally accorded to the dimensionless constants, but it has not made headway because of the basic difficulty that the current view of measurement precludes prior constraints on the possible range of values that any physical quantity—including these constants—can have. This difficulty does not exist in a constructive theory, and within the context of such a theory it is therefore proper to review the status of the dimensionless constants, and natural to seek theoretically determined values for them. Indeed one can see in a general way that in a constructive theory there would have to be constants having the kind of relation to measurement that the dimensionless constants have. I shall call such constants *scale-constants*.

3. Half-integral spin. The construction algorithm described in the last section was shown to generate levels of discrimination which we shall expect to be able to identify in existing physical theory. In fact the two simplest levels will be associated with spin vectors and space vectors respectively. The former consists of two and the latter of three independent (in my development—discriminable) components. (The places of time, and of relativistic invariance, in the hierarchy are very important and will be discussed later in this paper.)

The half-integral character of spin, which in several papers in this book has been regarded as the absolutely characteristic thing about the quantum-theoretic formalism, originates—I contend—in a mathematical relationship whose existence current theory has not led us to suspect. The hierarchy construction requires that operations of a given type at one level have an operation of a second type to produce the same effect at the next level. It is found that at the spin level there exist operations which have no single operation at the next level to correspond to them. They have to be replaced by two operations. There exist none that have to be replaced by three operations. It is this relationship which is usually interpreted by saying that the spin vector rotates π under a rotation through 2π of the space axes—giving half-integral spin. In current theory half-integral spin is a property of the only type of operator that would provide a two-spin state for the electron, and therefore depends on observation of spectral structure. I claim to deduce it.

I consider vectors of two elements and investigate the way in which order is preserved under discrimination operations. It is convenient to write vectors in columns and to represent the preservation of the order between the binary elements 0, 1 by writing a particular one of them in—say—the top position in the vector. The choice of which element we consider is not arbitrary once it has been decided which shall compose the null vector (and the 0 symbol was chosen for this purpose).

The discrimination operation B on the elements of a set S which was defined in section 2 may generate subsets S', S'', ..., of S which are closed under B. Thus for $x, y \in S$ we shall have then

$$x \neq y \rightarrow B(xy) \in S^{i} \qquad (3.1)$$
$$(\text{and} \quad B(xy) \rightarrow x \neq y \in S^{i})$$

for any i. It can easily be seen that the null vector or ordered set of zero elements cannot be a member of any S^{i}, as otherwise the above relation will not be invariant under change of i. (This result has been assumed in stating the numbers of entities at different levels.)

There is thus a distinction built into the discrimination algebra between the two symbols 0 and 1 which is unfamiliar from ordinary linear algebra, and which is vital to the representation of spin. The simplest level of the hierarchy contains vectors of two elements, and a sufficient representation at this level of the basic asymmetry of 0 and 1 in the discrimination calculus is secured by partitioning operators into two classes: those that have 1 in the first place (top, if we write the vectors as vertical columns) and those with 0 in the first place. Let us call the former class 'allowed', and the second 'disallowed'. We find that there exist disallowed operations which cannot be replaced by a single allowed operator. These require two allowed operations to perform the transformation that would have been produced by the given disallowed operator. There are no disallowed operators which require to be replaced by three or more allowed operators.

The operators which can be represented by two vectors, and which have a 1 in the top place provide the following transformations:

$$\underbrace{\begin{pmatrix}1\\0\end{pmatrix}\to\begin{pmatrix}0\\1\end{pmatrix}}_{\begin{pmatrix}1\\1\end{pmatrix}} \quad \underbrace{\begin{pmatrix}0\\1\end{pmatrix}\to\begin{pmatrix}1\\0\end{pmatrix}}_{\begin{pmatrix}1\\1\end{pmatrix}} \quad \underbrace{\begin{pmatrix}1\\1\end{pmatrix}\to\begin{pmatrix}0\\1\end{pmatrix}}_{\begin{pmatrix}1\\0\end{pmatrix}} \quad \underbrace{\begin{pmatrix}0\\1\end{pmatrix}\to\begin{pmatrix}1\\1\end{pmatrix}}_{\begin{pmatrix}1\\0\end{pmatrix}}. \tag{3.2}$$

The null vector is excluded.

By symmetry there are four which have a zero in the top place and which require two successive vector operations to effect them. Of these four we again reject two as being generated by the null vector, leaving two, namely:

$$\underbrace{\begin{pmatrix}1\\1\end{pmatrix}\quad\begin{pmatrix}1\\0\end{pmatrix}}_{\begin{pmatrix}0\\1\end{pmatrix}} \qquad \underbrace{\begin{pmatrix}1\\0\end{pmatrix}\quad\begin{pmatrix}1\\1\end{pmatrix}}_{\begin{pmatrix}0\\1\end{pmatrix}}. \tag{3.3}$$

When we look for pairs of allowed operations that will replace these, we find only one pair in each case, namely:

$$\underbrace{\begin{pmatrix}1\\1\end{pmatrix}\quad\begin{pmatrix}0\\1\end{pmatrix}\quad\begin{pmatrix}1\\0\end{pmatrix}}_{\begin{pmatrix}1\\0\end{pmatrix}\ \begin{pmatrix}1\\1\end{pmatrix}} \qquad \underbrace{\begin{pmatrix}1\\0\end{pmatrix}\quad\begin{pmatrix}0\\1\end{pmatrix}\quad\begin{pmatrix}1\\1\end{pmatrix}}_{\begin{pmatrix}1\\1\end{pmatrix}\ \begin{pmatrix}1\\0\end{pmatrix}}. \tag{3.4}$$

The foregoing analysis is expressed for convenience in terms of operators, but holds for the possibilities of replacement of vectors under the discrimination operation in any circumstances. We can generalize it as follows:

Let a set S be given of allowed vectors. Call this the *replacement subset*. Then

$$\mathbf{A}\mathbf{v} = \mathbf{a}+\mathbf{v} \tag{3.5}$$

for certain vectors $\mathbf{v}$ where $\mathbf{A}$ is a matrix and $\mathbf{a}$ is a member of the replacement subset. Thus one can choose $\mathbf{v}$ and define $\mathbf{v}$ in terms of expression (3.5), and consider the question of whether $\mathbf{a}$ is a member of S. It may happen that although $\mathbf{Av} \neq \mathbf{a}+\mathbf{v}$ for any $\mathbf{a}$, S, yet there exist two (or more) vectors in S ($\mathbf{a}$, $\mathbf{b}$ say for the case of two) such that

$$\mathbf{Av} = \mathbf{a}+\mathbf{b}+\mathbf{v}. \tag{3.6}$$

This will be possible as long as $\mathbf{a}+\mathbf{b}$ does not belong to S.

One can ask: what sort of replacement subsets give rise to such behaviour? For the level of n-vectors the smallest subset which will, by repeated additions, serve for the whole lot of $\mathbf{A}$ is a set of n linearly independent vectors, since sums for any number of these generate the whole space. In order to give a more general idea of the relation of allowed and disallowed vectors it has been necessary to use matrix transforms which are part of the record system of the hierarchy. Its use here is only illustrative since for this very simple case a complete solution has been obtained by enumeration of cases: it does not matter that the matrix transform is only a special case of a record system.

4. Spin as a dynamical concept. A large part of the structure of quantum mechanics has its origin in the belief that a satisfactory theory of the microscopic structure of matter needs to be expressed in terms of a mathematics which has a dynamical interpretation of a more general sort than that of the mathematics of classical mechanics, so that discrete (quantum jump) and continuous systems can be treated as special cases of the general formalism. In theories like that now being discussed this belief no longer applies, and therefore one must carefully compare the new proposals with the old to see which parts of the old have to reappear intact in the new and which were merely necessary to enable the old theory to conform to the belief which is being called into question.

It is useful to consider some observations of Pauli [3] in the paper in which he introduces the spin concept. His spin operators† satisfy

$$s_x s_y - s_y s_x = 2is_x \quad \text{etc.} \tag{4.1}$$

where

$$s_x^2+s_y^2+s_z^2 = 3 \tag{4.2}$$

and where the spin operators s_i are chosen in a particular way so that, in the special case where $\mathbf{s}$ has the value $\frac{1}{2}$, the equation becomes

$$s_x s_y = -s_y s_x = is_z \tag{4.3}$$

$$s_x^2 = s_y^2 = s_z^2 = 1.$$

† The s_μ are directly analogous to the vectors of (3.5) although printed as vector components. In a hierarchy, vector status is relative, depending on choice of level.

The representation of the s_i to achieve this result is

$$s_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Pauli remarks that in this special case equation (4.1) takes on the 'sharpened form' (4.3), so that a digital structure is achieved as a limiting case of one which can be related to conventional continuum mechanics. In achieving this result the half-integral spin appears, and the view of the digital as a limiting case makes it natural to interpret the spin vector as undergoing a rotation through an angle π when the space axes rotate through 2π, and this is indeed the conventional way of speaking.

If, however, we take the discrete structure as the primitive one, we have to re-examine the unity of the picture which Pauli presents. The success of Pauli's picture consisted in reconciling three requirements in one formalism.

(1) That the spin variables have two eigenvalues and could therefore explain the spectral observations.

(2) That the spin variables be incorporated into a quantum mechanics having the commutation relations which were at that date already established.

(3) That transformations of the spin variables be of a suitable mathematical form to be regarded as transformations of three-dimensional physical space. The resulting formalism had the half-integral spin as a consequence.

By contrast, the central advantage of the present theory is that it reverses this deductive sequence. The half-integral spin arises as a basic consequence of the method that has been used to set up an interpretable physical space of any sort—discrete in this case. Condition (1) is satisfied automatically. Condition (2) changes completely. The particular form of commutation relations are dictated by a coherent scheme of continuum mechanics which in the present approach has to be reconstructed piecemeal and which contains much that is arbitrary or historically accidental. This goes in particular for the use of the field of complex numbers as contrasted with the binary field 0, 1 in vector algebra. The one aspect of the coherence of the current mathematics of spinors and their commutation relations which seems to need more explaining away is their analogy with Jacobi's form of classical dynamics. It seems reasonable to expect that this analogy may be explicable on the basis of an extension of the de Rham theory to the topology of our discrimination calculus, but this will not be attempted in this paper.

The third of the requirements will be discussed in a later section of this paper.

5. Spin and dichotomous choice. There have been attempts to understand the spin calculus in terms of a principle of dichotomous choice which resemble the present approach in attaching fundamental significance to binary quantities. In particular Rosenfeld [(4)] develops an abstract 'dichotomic mathematics' or mathematics of dichotomous choices which gives him a justification for introducing spin vectors. He says:

When we have to distinguish between two possible states of some system, such as, e.g. the two states of different charge (proton and neutron) of a nucleon, we may conveniently characterize them by the eigenvalues, say $+1$ and -1, of some quantal variable. A discrimination between the two states in question then corresponds to a 'measurement' of the variable, such a measurement yielding the answer $+1$ or -1. In a matrix representation, such a *dichotomic variable* will appear as a Hermitean matrix with 2 rows and 2 columns, and will satisfy the equation

$$\tau^2 = 1.$$

From these conditions, it is easily deduced that the most general form for τ is

$$\tau = \begin{pmatrix} a & \sqrt{(1-a^2)}\,e^{-i\phi} \\ \sqrt{(1-a^2)}\,e^{i\phi} & -a \end{pmatrix}.$$

From this form, an interpretation of the dichotomic variables in terms of spatial directions is straightforward. Rosenfeld follows the method of establishing an interpretation for his new structure and then identifying that structure with existing physical concepts at each stage in its development, just so far as that identification is justified by the formal correspondence of the new structure with current theory. This is also my method. The difficulty with this method in either case is that you can never call a halt and say you have arrived. I am prepared to swallow this pill: Rosenfeld is not. In Rosenfeld's case a logical development from the idea of dichotomous choice is made to revert suddenly to the quite different ideas on which current physics is ultimately based, and you cannot do this. Indeed in general it is not possible to get support for a theory partly from each of two unrelated sets of ideas.†

† Even in law, where I am allowed to plead either that I did not kill X or that if I did, there were extenuating circumstances, neither plea is held to be strengthened by the existence of the other.

Rosenfeld attempts to avoid this trouble by clearly separating his dichotomic choice calculus from its application.

The correspondence here discussed between dichotomic variables and spatial directions is, in the general case, purely symbolical. But it acquires a real significance in the theory of spin, or intrinsic angular momentum, of the electron or nucleon. For the establishment of this concept the existence of a set of dichotomic variables forming a vector, together with the relations between them, is of essential importance.

The attempt does not really work though: the understanding of spin in terms of dichotomous choice *depends* upon the discreteness of the variables which represent that choice, and in this case a continuous application of a discrete calculus precludes the basis of understanding of that calculus.

My proposal regarding Rosenfeld's analysis of spin is not to reject it itself but to reject the possibility of suddenly jumping from it into the use of the modified current concepts. Such a proposal implies a programme along similar lines to the one I am describing in this paper.

6. Lorentz invariance in a discrete theory. If we accept the identification of spin with certain operators in the simplest levels of a discrimination hierarchy, then the identification of the three discriminable 2-vectors with space directions follows, This identification at first sight looks unnatural, though, as Rosenfeld also points out, remarkable. The feeling of unnaturalness arises, in fact, because one tends to think that one cannot use the language of dynamics at all without using all of it, and in particular if one speaks of the directions of space then the whole possibility of considering indefinitely large sets of points in that space must already exist. Instead, we have the following physical picture: any physical phenomenon is defined by some special restriction on, or structuring of, the discrimination process at some level, and if one wishes to know more detail about that phenomenon one does experiments and attempts to incorporate the knowledge one has gained by identifying more structure at that level. This is not necessarily possible—a fact which was first forced on our attention by the very breakdown of classical physics which led to quantum theory. In these circumstances one has to change to a new level with more elements. There are two effects that this change may have: first, a possibility of making new distinctions appears, and secondly, distinctions hitherto made may vanish.

The converse process of moving to simpler levels is more familiar. Any newly discovered detail in any physical phenomenon is always ultimately analysed—if it can be analysed at all—in terms of the concepts of mechanics and electrodynamics and these concepts derive their non-metrical characteristics from the algebraic structure of the simple hierarchy levels, to which one has to return recursively in any application of the hierarchy.

Since we reject attempts like that of Rosenfeld to reconcile conventional space-time with spin seen as an abstract principle of dichotomy it seems we must sacrifice some important insights. In particular there seems no place for Dirac's derivation of the spin vectors from a linear four-dimensional form of the wave equation, which is usually held to demonstrate a connexion between relativistic invariance and the quantum world.

I argue that no insights are sacrificed, but that on the contrary certain puzzles could be removed from the area of overlap of relativity and quantum theory by adopting the idea of physical space as a constructive space using the hierarchy model. My argument consists of the following steps.

(1) The connexion between 4-component spin matrices and the Lorentz group is accidental and limited to the common appearance of the number 4. The familiar treatment of Dirac [5] who showed that a 4-component wave equation with linear operators α_μ afforded a satisfactory representation of the Pauli spin variables, can be related to my present approach: Dirac's 4×4 matrices ρ_μ, which define a representation of the spin vectors in terms of the α_μ through

$$\alpha_1 = \rho_1\sigma_1, \quad \alpha_2 = \rho_1\sigma_2, \quad \alpha_3 = \rho_1\sigma_3, \quad \alpha_\mu = \rho_3$$

can be identified with the 4×4 matrix forms which I introduced earlier as a possible form of the necessary record system.

(2) There is no need to attempt to identify the dimension number of space-time with the order of 4-vectors in the spin theory, since in the discrimination algebra both appear, but in *a quite separate manner*.

(3) If Lorentz invariance has not got to be found a place in quantum theory then we no longer have to struggle with the extremely puzzling fact that the operational basis of relativity theory and hence the experimental reasons for imagining time as a dimension analogous to the space dimensions are so remote from those of quantum theory with its predominantly local, laboratory scale experimentation.

It is possible, consistently with the discrete approach to space that I am proposing, to regard the assimilation of time to space that underlies relativity theory as a mathematical device to procure a limiting velocity (namely the velocity of light) and hence to give a primary place to the dimensionless scale-constants (see section 2 of this paper) without a change of physical concepts at the most basic level. A brief account of this possibility has been given[2] and a fuller development of it is being written by Atkin and myself[6]. I shall not discuss it further now, its place in my present argument being only to indicate how to make the insights of relativity theory fit into the picture.

The picture in fact provided by the hierarchy constitutes a dynamic view of the acquisition of knowledge about any physical process, and it is natural to identify time with the sequence of stages through which the hierarchy passes. Obviously, it is important to keep this primary meaning for the word 'time' quite separate from the sophisticated concept of co-ordinate time. The reason it is so vital clearly to distinguish these two meanings is that very important deductions can be made from the theory as consequences of the conditions that have to be satisfied in advancing from the one time concept to the other. The treatment of spin which has been the subject of this paper can be seen as such a consequence.

The physical picture which emerges from my approach lacks the permanent objective background which the physicist expects to exist automatically. Each new investigation has to define its own starting point and construct its own spatial distribution of objects as it develops ('in time', if one wishes so to speak). Such a theory may have a very real advantage in dealing with high energy quantum systems where, for example, we are strongly impelled to think in terms of time-reversal—a concept which is practically inconceivable against a conventional invariant space-time background. In a theory which does not presuppose an invariant space-time background there will be an initial problem of constructing enough of such a background for any application that is being considered. One will have to ensure that one can fit the bits together. This requirement explains the importance of the scale constants in my approach. They are invariant for all applications of the theory and will therefore assume initial importance in interpretation, by contrast with their position in current theory where they are remote from natural interpretation. Further developments of this theory will require the discovery of quantities of an intermediate status which have some degree of invariance so as to make it plausible to interpret them as physical quantities. Then the conditions for them to be invariant prescribe the nature of their interpretation: they will be invariant in a certain context and that context will determine their interpretation.

7. Computing methods. The discrimination calculus results in a hierarchy in which at each level the points are constructed from structures (ordered subsets) at the previous level. One does not, however, begin afresh at each level because of the record system for relating structures at the different levels. This record system is easily considered as a digital computer memory, and since many of the properties of a discrimination hierarchy appear in a natural form when thought of as the operations of a computing system (this is specially true of the representation of physical time in a hierarchy) I conclude this paper with an idealized computer program to illustrate the use of the record system in relating levels.

The program is written in a much simplified form of TRAC (I begin by stating TRAC in four primitive terms and operations). TRAC itself is a high level programming language[(7), (8)] which is implemented and used at Cambridge Language Research Unit because of its logically primitive character and its applicability for purposes like the one I have now in hand.

Primitive Terms and Operations

1. *String*: an ordered set of characters or ciphers.
2. *Store*: set of ordered pairs each consisting of one string with one name. At any given time there is a one to one correspondence of strings and names.
3. *Call*: operation of drawing a string from store by specifying its name. The string also remains in store.
4. *Define*: operation of attaching a new string to a given name (and therefore of placing the string in store). A string defined with a given name replaces any string already in the store of that name.

Formal Rules

1. There exists a set $I = I(A, B, \ldots$ to h terms) of binary strings (strings consisting entirely of the binary units 0, 1), of equal length l, in store at a given time.
2. The names $a, b, \ldots$ to h terms, of the strings in I are binary strings all of length n.
3. A generating process is defined for constructing members of I. If P and Q (P and Q in I) are called, then a new string R is defined such that for all P, Q there exists a unique R.
4. R is of the same length as P and Q.

5. If and only if P has the same binary unit in each given place as Q has in that place, R is the null string consisting of zeros in each place.
6. A sequential process is begun by calling two non-null strings P, Q. We write the generation of R, $D(PQ) \rightarrow R$. (In the following rules 7, 8, 9 we specify the selection of the next string.)
7. The name $N(qr)$ of length l^2 is formed from the names q, r in such a way that each digit i_q in q and each digit j_r in r determines the choice of some digit ij in N, that way being independent of i and of j.
8. $N(qr)$ is the name of a member, P', of a second set of strings I'.
9. The set I' has as its elements strings which are ordered subsets of the $2^l - 1$ non-null elements of I, and which obey the same principle of generation as the elements of I.
10. The name of the new string s' in I is an ordered set of n digits of P' selected according to some consistent rule which I do not here specify.

The foregoing set of rules contains arbitrary elements in rules 3, 7, 10 which can be eliminated by completing the algorithms by randomizing over the choices left unspecified. A combination of rules 3 and 5 is sufficient to specify the generation operation as discrimination as defined in this paper and in the case of rule 3 the non-specificity is a formal necessity. The arbitrariness in rules 7 and 10 is real, however, and the actual choice of specific rules will determine the application of the program in detail. Nevertheless all the deductions made in this paper would remain true if the unspecified choices in rules 7 and 10 were made at random.

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